A quick route to sums of powers

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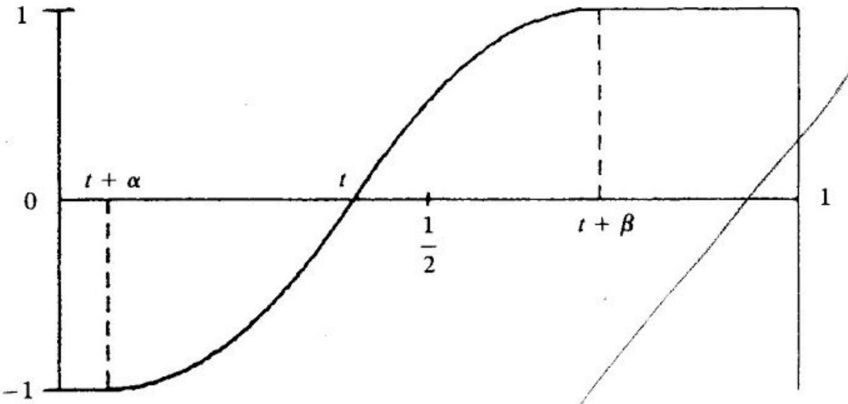

FIG. 5. An optimal function corresponding to $m = M > 4$ and $t \in (1/\sqrt{m}, 1/2)$.

EMARK. The idea of using asymmetric bounds for this type of problem apparently originated L. Hörmander [4].

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A QUICK ROUTE TO SUMS OF POWERS

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t is a remarkable fact that the polynomial for the sum of the rth powers of the integers

$$\sum_{v=1}^{n} v^r = a_1 n + a_2 n^2 + \cdots + a_{r+1} n^{r+1}$$

be expressed in terms of the first two sums

W. F. Edwards: I am Reader in Mathematical Biology in the University of Cambridge. An interest in the dations of statistics (manifest in my 1972 book *Likelihood*, Cambridge University Press) has led to historical rches in the origins of combinatorial theory, culminating in a book *Pascal's Arithmetical Triangle* (Griffin, Wycombe; in press), and thence to Bernoulli's *Ars conjectandi* and the work of Faulhaber.

n my spare time I am President of the C.U. Gliding Club and holder of the international Gold badge for ng (as is my wife also). I have contributed to the theory of cross-country soaring, which is a fascinating exercise plied mathematics.

$$\sum_{v=1}^{n} v = n(n+1)/2$$

and

$$\sum_{v=1}^{n} v^2 = n(n+1)(2n+1)/6,$$

a result ultimately traceable to the symmetry of the Bernoulli polynomials. The first example is, of course, the familiar yet striking relation

$$\sum_{v=1}^{n} v^3 = \left\{ \sum_{v=1}^{n} v \right\}^2;$$

the general result was proved by Jacobi (1834).

But perhaps even more striking than the result itself is the fact that it was known over two centuries before Jacobi's time, by the forgotten German mathematician Johann Faulhaber, in whose *Academia algebrae* (Augspurg, 1631) I found it in 1981 whilst pursuing a lead from James Bernoulli's *Ars conjectandi* (Basel, 1713). At the point where he introduces the polynomials (1) and gives a table of the coefficients (including, of course, the Bernoulli numbers, so-called by de Moivre), Bernoulli mentions the name of Faulhaber. It so happened that the one work of Faulhaber readily accessible to me was the copy of *Academia algebrae* belonging to Cambridge University, oddly enough once the property of Jacobi (though whether he acquired it before or after 1834 we cannot say).

Writing $\sum_{v=1}^{n} v^r$ as $\sum n^r$ for simplicity, we find that Faulhaber's polynomials are

$$\sum n^r (r \text{ even}) = \sum n^2 \cdot \left(b_1 + b_2 \sum n + b_3 \left(\sum n\right)^2 + \cdots b_{r/2} \left(\sum n\right)^{(r/2)-1} \right)$$

and

$$(2) \quad \sum n^r (r \text{ odd} \geqslant 3) = \left(\sum n\right)^2 \cdot \left(c_1 + c_2 \sum n + c_3 \left(\sum n\right)^2 + \cdots + c_{(r-1)/2} \left(\sum n\right)^{(r-3)/2} \right),$$

where of course the coefficients b_i and c_i differ with each r.

Faulhaber gave an algorithm for obtaining the coefficients c_i from the b_i for the preceding value of r, and a method for obtaining the b_i themselves. I have christened the forms (2) 'Faulhaber polynomials' (Edwards, 1982), and Schneider (1983) has given an account of his methods. The purpose of the present paper is to exhibit the matrix forms for the Faulhaber polynomials in a way analogous to the matrix forms for the polynomials (1) (Edwards, 1982). Apart from its intrinsic elegance this approach allows Faulhaber's algorithm for obtaining the c_i from the b_i to be easily understood.

Consider the expansion of $[x(x+1)]^r - [x(x-1)]^r$, and apply to it Pascal's method of writing the identity successively for $x = 1, 2, 3, \ldots n$ and summing, as suggested by Tits (1923). We obtain (in the Σ notation introduced above)

$$(3) \qquad [n(n+1)]^r = 2\left[r \sum n^{2r-1} + \binom{r}{3} \sum n^{2r-3} + \binom{r}{5} \sum n^{2r-5} + \cdots \right],$$

which may be written in matrix form with rows for $r = 2, 3, 4, \ldots,$

$$(4) \qquad \begin{pmatrix} [n(n+1)]^2 \\ [n(n+1)]^3 \\ [n(n+1)]^4 \\ [n(n+1)]^5 \\ \cdot \end{pmatrix} = 2 \begin{pmatrix} 2 & & & & \\ 1 & 3 & & 0 & \\ 0 & 4 & 4 & & \\ 0 & 1 & 10 & 5 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \sum n^3 \\ \sum n^5 \\ \sum n^7 \\ \sum n^9 \\ \cdot \end{pmatrix}$$

in which each row of the matrix is the corresponding row of Pascal's triangle with every other coefficient omitted. Writing $u = n(n+1)$ and solving (4) for the sums of the odd powers, we have

$$\begin{pmatrix} \sum n^3 \\ \sum n^5 \\ \sum n^7 \\ \sum n^9 \\ \cdot \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & & & & \\ 1 & 3 & & 0 & \\ 0 & 4 & 4 & & \\ 0 & 1 & 10 & 5 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}^{-1} \begin{pmatrix} u^2 \\ u^3 \\ u^4 \\ u^5 \\ \cdot \end{pmatrix}, \tag{5}$$

which is the complete solution for the 'odd' Faulhaber polynomials since $u = 2\Sigma n$. In particular, u^2 is a factor of every polynomial, this proving the form given in (2).

Call the matrix of (4) $\mathbf{F} = \{f_{ij}\}$, then

$$f_{ij} = \binom{i+1}{2(i-j)+1}, \tag{6}$$

or zero for all values of i and j which do not define binomial coefficients. The matrix

$$\mathbf{F}^{-1} = \begin{pmatrix} 1/2 & & & & \\ -1/6 & 1/3 & & 0 & \\ 1/6 & -1/3 & 1/4 & & \\ -3/10 & 3/5 & -1/2 & 1/5 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{7}$$

thus leads to all the coefficients of the odd Faulhaber polynomials, when due allowance is made for the factor 2 in $u = 2\Sigma n$.

Tits also treated the even polynomials, applying Pascal's method to the expansion of $x^r(x+1)^{r+1} - x^{r+1}(x-1)^r$ and using (3) to remove the odd powers. The result, written in matrix form, is

$$\begin{pmatrix} \sum n^2 \\ \sum n^4 \\ \sum n^6 \\ \sum n^8 \\ \cdot \end{pmatrix} = \frac{1}{2}(2n+1) \cdot \begin{pmatrix} 3 & & & & \\ 1 & 5 & & 0 & \\ 0 & 5 & 7 & & \\ 0 & 1 & 14 & 9 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}^{-1} \begin{pmatrix} u \\ u^2 \\ u^3 \\ u^4 \\ \cdot \end{pmatrix}. \tag{8}$$

Call the matrix of (8) $\mathbf{G} = \{g_{ij}\}$, then

$$g_{ij} = \binom{i+1}{2(i-j)+1} + \binom{i}{2(i-j)+1}, \tag{9}$$

undefined binomial coefficients again being replaced by zeros. It will be seen that

$$\mathbf{G} = \mathbf{F} + \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot \\ 0 & & & & \\ 0 & & & \mathbf{F} & \\ 0 & & & & \\ \cdot & & & & \end{pmatrix}. \tag{10}$$

The matrix

$$(11) \qquad \mathbf{G}^{-1} = \begin{pmatrix} 1/3 & & & & \\ -1/15 & 1/5 & & 0 & \\ 1/21 & -1/7 & 1/7 & & \\ -1/15 & 1/5 & -2/9 & 1/9 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

thus leads to all the coefficients of the even Faulhaber polynomials.

Faulhaber knew, in essence, how to obtain $\mathbf{F}^{-1}$ from $\mathbf{G}^{-1}$. We may discover his algorithm as follows.

Take **F** and divide the elements of each column by the diagonal element in that column and thus write

$$(12) \qquad \mathbf{F} = \begin{pmatrix} 2 & & & & \\ 1 & 3 & & 0 & \\ 0 & 4 & 4 & & \\ 0 & 1 & 10 & 5 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 2 & & & & \\ & 3 & & 0 & \\ & & 4 & & \\ 0 & & & 5 & \\ & & & & \cdot \end{pmatrix} \begin{pmatrix} 1 & & & & \\ 1/3 & 1 & & 0 & \\ 0 & 1 & 1 & & \\ 0 & 1/5 & 2 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Now take **G** and divide the elements of each row by the diagonal element in that row and thus write

$$(13) \qquad \mathbf{G} = \begin{pmatrix} 3 & & & & \\ 1 & 5 & & 0 & \\ 0 & 5 & 7 & & \\ 0 & 1 & 14 & 9 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ 1/3 & 1 & & 0 & \\ 0 & 1 & 1 & & \\ 0 & 1/5 & 2 & 1 & \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} 3 & & & & \\ & 5 & & 0 & \\ & & 7 & & \\ 0 & & & 9 & \\ & & & & \cdot \end{pmatrix}.$$

Comparing (12) and (13) we see that the derived matrices are identical, an identity which, when analysed using (6) and (9), rests on a simple (and uninteresting) identity involving binomial coefficients.

Now write

$$\mathbf{X} = \begin{pmatrix} 2 & & & & \\ & 3 & & 0 & \\ & & 4 & & \\ 0 & & & 5 & \\ & & & & \cdot \end{pmatrix}$$

and

$$\mathbf{Y} = \begin{pmatrix} 3 & & & & \\ & 5 & & 0 & \\ & & 7 & & \\ 0 & & & 9 & \\ & & & & \cdot \end{pmatrix}$$

and we have $\mathbf{X}^{-1}\mathbf{F} = \mathbf{G}\mathbf{Y}^{-1}$ or, inverting,

$$\mathbf{F}^{-1}\mathbf{X} = \mathbf{Y}\mathbf{G}^{-1}$$

and thus

$$(14) \qquad \mathbf{F}^{-1} = \mathbf{Y}\mathbf{G}^{-1}\mathbf{X}^{-1}.$$

It is easily seen that premultiplication of $\mathbf{G}^{-1}$ by **Y** and postmultiplication by $\mathbf{X}^{-1}$ amounts to multiplying the rows of $\mathbf{G}^{-1}$ by $3, 5, 7, 9, \ldots$, respectively, and dividing the columns by $2, 3, 4, 5, \ldots$, respectively. Thus to obtain the jth coefficient in the ith row of $\mathbf{F}^{-1}$ we take the corresponding coefficient in $\mathbf{G}^{-1}$, multiply by $(2i + 1)$ and divide by $(j + 1)$.

For example, when $i = 4$ we multiply the four coefficients in the fourth row of $\mathbf{G}^{-1}$,

$$-\frac{1}{15}, \frac{1}{5}, -\frac{2}{9}, \frac{1}{9},$$

by

$$\frac{9}{2}, \frac{9}{3}, \frac{9}{4}, \frac{9}{5},$$

to obtain the fourth row of $\mathbf{F}^{-1}$,

$$-\frac{3}{10}, \frac{3}{5}, -\frac{1}{2}, \frac{1}{5}.$$

Thus the polynomial

$$\sum n^8 = \frac{1}{2}(2n + 1)\left(-\frac{1}{15}u + \frac{1}{5}u^2 - \frac{2}{9}u^3 + \frac{1}{9}u^4\right)$$

leads to

$$\sum n^9 = \frac{1}{2}\left(-\frac{3}{10}u^2 + \frac{3}{5}u^3 - \frac{1}{2}u^4 + \frac{1}{5}u^5\right).$$

Faulhaber's actual algorithm is different because we have worked with $u = 2\Sigma n$ rather than Σn, but the difference is, of course, trivial.

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Added in proof: Of related interest is B. L. Burrows and R. F. Talbot, Sums of powers of integers, this MONTHLY, 91 (1984) 394–403.

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MISCELLANEA

There is now, and there always will be room in the world for good mathematicians of every grade of logical precision. It is almost equally important that the small band whose chief interest lies in accuracy and rigor should not make the mistake of despising the broader though less accurate work of the great mass of their colleagues, as that the latter should not attempt to shake themselves wholly free from the restraint the former would put upon them.

—Maxime Bôcher, Bull. Amer. Math. Soc., 10 (1904) 135.