# From twin theorems on matrices obtainable from the Pascal triangle to the problem of sums of successive integers extended to any arithmetic progression

by Giorgio Pietrocola, May-July 2019

Summary:
In the first part there are two demonstrations on how to obtain polynomials corresponding to the sum of powers of successive integers: the theorem "1A" for sums from 1 to n and the theorem "1B" for sums from 0 to n-1.
It is then shown that the matrix found can be expressed in the traditional form indicated by the so-called Faulhaber's formula.

In the second part, more generally, the result is extended to any arithmetic progression thus generalizing Faulhaber's formula relating to sums of powers of successive integers.

Warning:
For purely illustrative reasons, the vectors will often be represented with six components (m = 6) and the square matrices will be 6 rows and six columns. In the symbolism adopted, the number m of components is not specified because m can be any positive integer that one is free to set at will.

## 1.First part: The twin theorems

### 1) Theorem "1A"

### On the polynomials corresponding to the sum of powers of successive integers from 1 to n

Notation:

$$\vec{V}(i) = \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix} \quad i\vec{V}(i) = \begin{pmatrix} i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{pmatrix} \quad \sum_{i=1}^{n} \vec{V}(i) = \vec{S}_1(n) = \begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^3 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^5 \end{pmatrix}$$

$$\overline{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{pmatrix}$$

We start from the following identity deriving from the development of the power of the binomial:

$$\begin{pmatrix} i-(i-1) \\ i^2-(i-1)^2 \\ i^3-(i-1)^3 \\ i^4-(i-1)^4 \\ i^5-(i-1)^5 \\ i^6-(i-1)^6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1+2i \\ 1-3i+3i^2 \\ -1+4i-6i^2+4i^3 \\ 1-5i+10i^2-10i^3+5i^4 \\ -1+6i-15i^2+20i^3-15i^4+6i^5 \end{pmatrix}$$

Using the vectors defined at the beginning and taking into account the product rows by column, the previous one becomes:

$$\begin{pmatrix} i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{pmatrix} - \begin{pmatrix} (i-1) \\ (i-1)^2 \\ (i-1)^3 \\ (i-1)^4 \\ (i-1)^5 \\ (i-1)^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix}$$

indicating the matrix with alternating signs, easily obtainable from Pascal's triangle as indicated at the beginning, we can summarize the identity by writing:

$$i\vec{V}(i) - (i-1)\vec{V}(i-1) = \overline{A}\vec{V}(i)$$

adding member to member for the variables from 1 to n we obtain:

$$\sum_{i=1}^{n}(i\vec{V}(i) - (i-1)\vec{V}(i-1)) = \sum_{i=1}^{n}\overline{A}\vec{V}(i)$$

By developing the sum to the first member, almost all the terms except the first and the last (telescopic effect) are simplified two by two and by collecting the common factor matrix of the second member, we obtain:

$$n\vec{V}(n) - (0)\vec{V}(0) = \overline{A}\sum_{i=1}^{n}\vec{V}(i)$$

Omitting the vector subtracted from null components and replacing the sum of vectors with the initially defined vector we obtain:

$$n\vec{V}(n) = \overline{A}\vec{S}_1(n)$$

finally to explicate the vector S multiply both members of the equation on the left by the inverse matrix of A marked (existing because inverse of a triangular matrix with determinant m!, non-zero product of the main diagonal):

$$\vec{S}_1(n) = \overline{A}^{-1} n\vec{V}(n)$$

which solves the traditional problem of the sum of powers of successive integers

**1.1.1) Example in the case of seven components (m = 7):**

$$\begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^3 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^5 \\ \sum_{i=1}^{n} i^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 & 0 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 \end{pmatrix}^{-1} \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \end{pmatrix}$$

**1.1.2) Example in the case of eleven components (m = 11):**

$$\begin{pmatrix} \sum_{i=1}^{n} i^0 \\ \sum_{i=1}^{n} i^1 \\ \sum_{i=1}^{n} i^2 \\ \sum_{i=1}^{n} i^3 \\ \sum_{i=1}^{n} i^4 \\ \sum_{i=1}^{n} i^5 \\ \sum_{i=1}^{n} i^6 \\ \sum_{i=1}^{n} i^7 \\ \sum_{i=1}^{n} i^8 \\ \sum_{i=1}^{n} i^9 \\ \sum_{i=1}^{n} i^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & \frac{1}{2} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & \frac{1}{2} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{7} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & 0 & -\frac{7}{24} & 0 & \frac{7}{12} & \frac{1}{2} & \frac{1}{8} & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{2}{9} & 0 & -\frac{7}{15} & 0 & \frac{2}{3} & \frac{1}{2} & \frac{1}{9} & 0 & 0 \\ 0 & -\frac{3}{20} & 0 & \frac{1}{2} & 0 & -\frac{7}{10} & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{10} & 0 \\ \frac{5}{66} & 0 & -\frac{1}{2} & 0 & 1 & 0 & -1 & 0 & \frac{5}{6} & \frac{1}{2} & \frac{1}{11} \end{pmatrix} * \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{pmatrix}$$

i.e. running the product row by column:

$$\sum_{i=1}^{n} i^0 = n$$

$$\sum_{i=1}^{n} i^1 = \frac{1}{2}n + \frac{1}{2}n^2$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$$

$$\sum_{i=1}^{n} i^4 = -\frac{1}{30}n + \frac{1}{3}n^3 + \frac{1}{2}n^4 + \frac{1}{5}n^5$$

$$\sum_{i=1}^{n} i^5 = -\frac{1}{12}n^2 + \frac{5}{12}n^4 + \frac{1}{2}n^5 + \frac{1}{6}n^6$$

$$\sum_{i=1}^{n} i^6 = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 + \frac{1}{2}n^6 + \frac{1}{7}n^7$$

$$\sum_{i=1}^{n} i^7 = \frac{1}{12}n^2 - \frac{7}{24}n^4 + \frac{7}{12}n^6 + \frac{1}{2}n^7 + \frac{1}{8}n^8$$

$$\sum_{i=1}^{n} i^8 = -\frac{1}{30}n + \frac{2}{9}n^3 - \frac{7}{15}n^5 + \frac{2}{3}n^7 + \frac{1}{2}n^8 + \frac{1}{9}n^9$$

$$\sum_{i=1}^{n} i^9 = -\frac{3}{20}n^2 + \frac{1}{2}n^4 - \frac{7}{10}n^6 + \frac{3}{4}n^8 + \frac{1}{2}n^9 + \frac{1}{10}n^{10}$$

$$\sum_{i=1}^{n} i^{10} = \frac{5}{66}n - \frac{1}{2}n^3 + n^5 - n^7 + \frac{5}{6}n^9 + \frac{1}{2}n^{10} + \frac{1}{11}n^{11}$$

These are the polynomials that were published in Jacob Bernoulli's book Ars Conjectandi in 1713 a few years after the author's death.

## 1.2) "1B" theorem
## On polynomials corresponding to the sum of powers of successive integers from 0 to n-1

| Adopted notation (setting $0^0=1$): |
|---|

$$\vec{V}(i) = \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix} \quad i\vec{V}(i) = \begin{pmatrix} i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{pmatrix} \quad \sum_{i=0}^{n-1} \vec{V}(i) = \vec{S}_0(n) = \begin{pmatrix} \sum_{i=0}^{n-1} i^0 \\ \sum_{i=0}^{n-1} i^1 \\ \sum_{i=0}^{n-1} i^2 \\ \sum_{i=0}^{n-1} i^3 \\ \sum_{i=0}^{n-1} i^4 \\ \sum_{i=0}^{n-1} i^5 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{pmatrix}$$

We start from the following identity between two vectors deriving from the development of the power of the binomial:

$$\begin{pmatrix} (1+i)-i \\ (1+i)^2-i^2 \\ (1+i)^3-i^3 \\ (1+i)^4-i^4 \\ (1+i)^5-i^5 \\ (1+i)^6-i^6 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+2i \\ 1+3i+3i^2 \\ 1+4i+6i^2+4i^3 \\ 1+5i+10i^2+10i^3+5i^4 \\ 1+6i+15i^2+20i^3+15i^4+6i^5 \end{pmatrix}$$

Using the vectors defined at the beginning and taking into account the product rows by column, the previous one becomes:

$$\begin{pmatrix} (1+i) \\ (1+i)^2 \\ (1+i)^3 \\ (1+i)^4 \\ (1+i)^5 \\ (1+i)^6 \end{pmatrix} - \begin{pmatrix} i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix}$$

indicating with A the matrix obtainable from Pascal's triangle excluding the last element of each row, we can summarize the identity by writing:

$$(1+i)\vec{V}(1+i) - i\vec{V}(i) = A\vec{V}(i)$$

adding member to member for the variables from 0 to n-1 we obtain:

$$\sum_{i=0}^{n-1}((1+i)\vec{V}(1+i)-i\vec{V}(i))=\sum_{i=0}^{n-1}A\vec{V}(i)$$

By developing the sum to the first member, almost all the terms except the first and the last (telescopic effect) are simplified two by two and by collecting the factor matrix common to the second member, we obtain:

$$n\vec{V}(n)-(0)\vec{V}(0)=A\sum_{i=0}^{n-1}\vec{V}(i)$$

Omitting the vector subtracted from null components and replacing the sum of vectors with the initially defined vector we obtain:

$$n\vec{V}(n)=A\vec{S}_0(n)$$

finally to explicate the vector S, both members of the equation on the left multiply by the inverse matrix of A (existing because A is a triangular *mxm* matrix with the product of the diagonal m!, not null):

$$\vec{S}_0(n)=A^{-1}n\vec{V}(n) \qquad (1)$$

the problem of the sum of powers of successive integers is thus solved in general

1.2.1) Example in the case of m = 7 components:

$$\begin{pmatrix} \sum_{k=0}^{n-1} k^0 \\ \sum_{k=0}^{n-1} k^1 \\ \sum_{k=0}^{n-1} k^2 \\ \sum_{k=0}^{n-1} k^3 \\ \sum_{k=0}^{n-1} k^4 \\ \sum_{k=0}^{n-1} k^5 \\ \sum_{k=0}^{n-1} k^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 \end{pmatrix}^{-1} \cdot \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \end{pmatrix}$$

calculating the inverse matrix:

$$\begin{pmatrix} \sum_{k=0}^{n-1} k^0 \\ \sum_{k=0}^{n-1} k^1 \\ \sum_{k=0}^{n-1} k^2 \\ \sum_{k=0}^{n-1} k^3 \\ \sum_{k=0}^{n-1} k^4 \\ \sum_{k=0}^{n-1} k^5 \\ \sum_{k=0}^{n-1} k^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{6} & -\frac{1}{2} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ -\frac{1}{30} & 0 & \frac{1}{3} & -\frac{1}{2} & \frac{1}{5} & 0 & 0 \\ 0 & -\frac{1}{12} & 0 & \frac{5}{12} & -\frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{42} & 0 & -\frac{1}{6} & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{7} \end{pmatrix} \cdot \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \end{pmatrix}$$

developing the product row by column:

$$\sum_{i=0}^{n-1} i^0 = n$$
$$\sum_{i=0}^{n-1} i^1 = -\frac{1}{2}n + \frac{1}{2}n^2$$
$$\sum_{i=0}^{n-1} i^2 = \frac{1}{6}n - \frac{1}{2}n^2 + \frac{1}{3}n^3$$
$$\sum_{i=0}^{n-1} i^3 = \frac{1}{4}n^2 - \frac{1}{2}n^3 + \frac{1}{4}n^4$$
$$\sum_{i=0}^{n-1} i^4 = -\frac{1}{30}n + \frac{1}{3}n^3 - \frac{1}{2}n^4 + \frac{1}{5}n^5$$
$$\sum_{i=0}^{n-1} i^5 = -\frac{1}{12}n^2 + \frac{5}{12}n^4 - \frac{1}{2}n^5 + \frac{1}{6}n^6$$
$$\sum_{i=0}^{n-1} i^6 = \frac{1}{42}n - \frac{1}{6}n^3 + \frac{1}{2}n^5 - \frac{1}{2}n^6 + \frac{1}{7}n^7$$

## 1.3) Proof of Faulhaber's formula

We will show that the inverse matrix of A can be presented in the classic form.

For example in the case of six components (m = 5) we will show that:

$$
\begin{pmatrix}
1\frac{1}{1}B_0 & 0 & 0 & 0 & 0 & 0 \\
2\frac{1}{2}B_1 & 1\frac{1}{2}B_0 & 0 & 0 & 0 & 0 \\
3\frac{1}{3}B_2 & 3\frac{1}{3}B_1 & 1\frac{1}{3}B_0 & 0 & 0 & 0 \\
4\frac{1}{4}B_3 & 6\frac{1}{4}B_2 & 4\frac{1}{4}B_1 & 1\frac{1}{4}B_0 & 0 & 0 \\
5\frac{1}{5}B_4 & 10\frac{1}{5}B_3 & 10\frac{1}{5}B_2 & 5\frac{1}{5}B_1 & 1\frac{1}{5}B_0 & 0 \\
6\frac{1}{6}B_5 & 15\frac{1}{6}B_4 & 20\frac{1}{6}B_3 & 15\frac{1}{6}B_2 & 6\frac{1}{6}B_1 & 1\frac{1}{6}B_0
\end{pmatrix} \cdot
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 0 \\
1 & 6 & 15 & 20 & 15 & 6
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

We will show that the product row by column of the first by the second (and vice versa) give the neutral element of the product between matrices. The thesis will follow from the uniqueness of the inverse matrix. We proceed by induction. In the case m = 0 (1 row and a column) the result is B0 which is 1.

Since as m increases, the triangular matrices each time incorporate the previous results by completing the longest row with 0 and essentially adding only the last row, it will suffice to show that in the generic case m (m + 1 rows and columns) the last row is made up of all 0 except 1 at the end.

Multiplying the m-th row by the j-th column (starting from 0) and neglecting the null addenda due to the triangularity of the matrices, we obtain:

$$\sum_{k=j}^{m} \frac{1}{m+1} \binom{m+1}{k+1} B_{m-k} \binom{k+1}{j}$$

By highlighting the factor independent of c h and expressing the binomial coefficients by means of factorial we have:

$$\frac{1}{m+1} \sum_{k=j}^{m} \frac{(m+1)!}{(k+1)!(m-k)!} \frac{(k+1)!}{j!(k+1-j)!} B_{m-k}$$

changing the order of the factors and using the invariant property of the division we obtain:

$$\frac{1}{m+1} \sum_{k=j}^{m} \frac{(m-j+1)!}{(k+1-j)!(m-k)!} \frac{(m+1)!}{j!(m+1-j)!} B_{m-k}$$

expressing in the form of a binomial coefficient and highlighting what does not depend on k:

$$\frac{1}{m+1} \binom{m+1}{j} \sum_{k=j}^{m} \binom{m-j+1}{m-k} B_{m-k}$$

For the known properties of Bernoulli numbers, which can also be deduced from the product between the m-row of A and the first column of its inverse, the elements of the last row from j=0 to j=m-1, the summation is zero while for j=m the sum is reduced to a single addend equal to $B_0$ = 1. Since 1 is also the factor outside the summation, the last row corresponds precisely to the last of the neutral element.

Of course, by reversing the order of factors, for similar reasons, the same result is achieved:

$$\sum_{k=j}^{m} \frac{1}{k+1} \binom{m+1}{k} \binom{k+1}{j+1} B_{k-j}$$

$$\sum_{k=j}^{m} \frac{1}{k+1} \frac{(m+1)!}{k!(m+1-k)!} \frac{(k+1)!}{(j+1)!(k-j)!} B_{k-j}$$

$$\frac{1}{j+1} \sum_{k=j}^{m} \frac{(m-j+1)!}{(k-j)!(m+1-k)!} \frac{(m+1)!}{(m-j+1)!j!} B_{k-j}$$

$$\frac{1}{j+1} \binom{m+1}{j} \sum_{k=j}^{m} \binom{m-j+1}{k-j} B_{k-j}$$

**Second part: Further details**

## 2.1) The Abelian group of powers of T

We will denote by T the triangular matrix corresponding to the Pascal triangle.

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix}$$

For the development of the power of the binomial we have:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix} = \begin{pmatrix} (1+i)^0 \\ (1+i)^1 \\ (1+i)^2 \\ (1+i)^3 \\ (1+i)^4 \\ (1+i)^5 \end{pmatrix}$$

which can be expressed as

$$T\vec{V}(i) = \vec{V}(1+i)$$

therefore multiplication by T increases the basis of the power vector by one unit. By repeating the operation the overall increase will be 2 units and therefore, for the associative property, it must be $TT\mathbf{V}(i)=T^2\mathbf{V}(i)=\mathbf{V}(i+2)$

More generally:

$$\begin{pmatrix} 1 \\ h+i \\ h^2+2hi+i^2 \\ h^3+3h^2i+3hi^2+i^3 \\ h^4+4h^3i+6h^2i^2+4hi^3+i^4 \\ h^5+5h^4i+10h^3i^2+10h^2i^3+5hi^4+i^5 \end{pmatrix} = \begin{pmatrix} (h+i)^0 \\ (h+i)^1 \\ (h+i)^2 \\ (h+i)^3 \\ (h+i)^4 \\ (h+i)^5 \end{pmatrix}$$

equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h & 1 & 0 & 0 & 0 & 0 \\ h^2 & 2h & 1 & 0 & 0 & 0 \\ h^3 & 3h^2 & 3h & 1 & 0 & 0 \\ h^4 & 4h^3 & 6h^2 & 4h & 1 & 0 \\ h^5 & 5h^4 & 10h^3 & 10h^2 & 5h & 1 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix} = \begin{pmatrix} (h+i)^0 \\ (h+i)^1 \\ (h+i)^2 \\ (h+i)^3 \\ (h+i)^4 \\ (h+i)^5 \end{pmatrix}$$

expressible as:

$$T^h \vec{V}(i) = \vec{V}(h+i)$$

therefore the multiplication by $T^h$ increases the basis of the power vector of h. Therefore the set of powers of T with not only integer exponents (but also rational, real or complex) with the composition operation constitute an Abelian group isomorphic to that of the ordinary addition that they induce on the basis of the vectors V for which they multiply . Referring to the induced sums, it is found that

$$T^0 = I \ , \quad T^h T^{-h} = I \ , \quad T^a T^b = T^b T^a = T^{a+b}$$
$$(T^a T^b) T^c = T^a (T^b T^c) = T^{a+b+c}$$

Note that this way

$$T^h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h & 1 & 0 & 0 & 0 & 0 \\ h^2 & 2h & 1 & 0 & 0 & 0 \\ h^3 & 3h^2 & 3h & 1 & 0 & 0 \\ h^4 & 4h^3 & 6h^2 & 4h & 1 & 0 \\ h^5 & 5h^4 & 10h^3 & 10h^2 & 5h & 1 \end{pmatrix}$$

for example $T^{10}$ can be calculated without having to repeat the product many times row by column.

## 2.2) "1C" theorem

On polynomials corresponding to the sum of powers of successive integers from *h* a *h+n-1*

We extend the notation already adopted in the two previous theorems:

$$\sum_{i=h}^{h+n-1} \vec{V}(i) = \vec{S}_h(n) = \begin{pmatrix} \sum_{i=h}^{h+n-1} i^0 \\ \sum_{i=h}^{h+n-1} i^1 \\ \sum_{i=h}^{h+n-1} i^2 \\ \sum_{i=h}^{h+n-1} i^3 \\ \sum_{i=h}^{h+n-1} i^4 \\ \sum_{i=h}^{h+n-1} i^5 \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{n-1} (h+i)^0 \\ \sum_{i=0}^{n-1} (h+i)^1 \\ \sum_{i=0}^{n-1} (h+i)^2 \\ \sum_{i=0}^{n-1} (h+i)^3 \\ \sum_{i=0}^{n-1} (h+i)^4 \\ \sum_{i=0}^{n-1} (h+i)^5 \end{pmatrix} = \sum_{i=0}^{n-1} \vec{V}(h+i)$$

For our proof we can start from the theorem 1B which establishes:

$$\vec{S}_0(n) = A^{-1} n \vec{V}(n)$$

multiplying the two members on the left by $T^h$, at the first member the multiplication is distributed to the addends V (i) which become V (h + i) and therefore

$$\boxed{\vec{S}_h(n) = T^h A^{-1} n \vec{V}(n)}$$

**2.2.1) Corollary**

note, as a corollary, that for h = 1 it turns out

$$\vec{S}_1(n) = T A^{-1} n \vec{V}(n)$$

keeping in mind the theorem 1A it is proved that

$$\overline{A} = TA^{-1}$$

## 2.2.2) Example with h=-9 m=4

Calculating $T^{-9}A^{-1}$:

$$\begin{pmatrix} \sum_{i=-9}^{n-10} i^0 \\ \sum_{i=-9}^{n-10} i^1 \\ \sum_{i=-9}^{n-10} i^2 \\ \sum_{i=-9}^{n-10} i^3 \\ \sum_{i=-9}^{n-10} i^4 \\ \sum_{i=-9}^{n-10} i^5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{19}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{541}{6} & -\frac{19}{2} & \frac{1}{3} & 0 & 0 & 0 \\ -855 & \frac{541}{4} & -\frac{19}{2} & \frac{1}{4} & 0 & 0 \\ \frac{242999}{30} & -1710 & \frac{541}{3} & -\frac{19}{2} & \frac{1}{5} & 0 \\ -76665 & \frac{242999}{12} & -2850 & \frac{2705}{12} & -\frac{19}{2} & \frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \end{pmatrix}$$

and in equivalent form:

$$\sum_{i=-9}^{n-10} i^0 = n$$

$$\sum_{i=-9}^{n-10} i^1 = -\frac{19}{2}n + \frac{1}{2}n^2$$

$$\sum_{i=-9}^{n-10} i^2 = \frac{541}{6}n - \frac{19}{2}n^2 + \frac{1}{3}n^3$$

$$\sum_{i=-9}^{n-10} i^3 = -855n - \frac{541}{4}n^2 - \frac{19}{2}n^3 + \frac{1}{4}n^4$$

$$\sum_{i=-9}^{n-10} i^4 = \frac{242999}{30}n - 1710n^2 + \frac{541}{3}n^3 - \frac{19}{2}n^4 + \frac{1}{5}n^5$$

$$\sum_{i=-9}^{n-10} i^5 = -76665n + \frac{242999}{12}n^2 - 2850n^3 + \frac{2705}{12}n^4 - \frac{19}{2}n^5 + \frac{1}{6}n^6$$

Example with h=e m=4

$$\begin{pmatrix} \sum_{i=0}^{n-1}(e+i)^0 \\ \sum_{i=0}^{n-1}(e+i)^1 \\ \sum_{i=0}^{n-1}(e+i)^2 \\ \sum_{i=0}^{n-1}(e+i)^3 \end{pmatrix} = T^e A^{-1} n \vec{V}(n) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ e & 1 & 0 & 0 \\ e^2 & 2e & 1 & 0 \\ e^3 & 3e^2 & 3e & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} n\vec{V}(n) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ (e-\frac{1}{2}) & \frac{1}{2} & 0 & 0 \\ (e^2-e+\frac{1}{6}) & (e-\frac{1}{2}) & \frac{1}{3} & 0 \\ (e^3-\frac{3}{2}e^2+\frac{1}{2}e) & \frac{3}{2}(e^2-e+\frac{1}{6}) & (e-\frac{1}{2}) & \frac{1}{4} \end{pmatrix} \begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \end{pmatrix} =$$

$$= \begin{pmatrix} n \\ (e-\frac{1}{2})n+\frac{1}{2}n^2 \\ (e^2-e+\frac{1}{6})n+(e-\frac{1}{2})n^2+\frac{1}{3}n^3 \\ (e^3-\frac{3}{2}e^2+\frac{1}{2}e)n+\frac{3}{2}(e^2-e+\frac{1}{6})n^2+(e-\frac{1}{2})n^3+\frac{1}{4}n^4 \end{pmatrix}$$

Note that the quantities in parentheses in the polynomials in n are the Bernoulli polynomials calculated in e.

### 2.3 Infinite Bernoulli sequences.

We denote by **B**(h) the array corresponding to the first column of $T^hA^{-1}$. Since the first column of $A^{-1}$ is the vector **B** of the Bernoulli numbers, it results:

$$\vec{B}(n) = T^h \vec{B}$$

$$\vec{B}(-1) = \left(1, -\frac{3}{2}, \frac{13}{6}, -3, \frac{119}{30}, -5, \frac{253}{42}, -7, \frac{239}{30}, -9, \frac{665}{66}, -11, \frac{32069}{2730}, \ldots\right)$$
$$\vec{B}(0) = \left(1, -\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \ldots\right)$$
$$\vec{B}(1) = \left(1, \frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, 0, \frac{1}{30}, 0, \frac{5}{66}, 0, -\frac{691}{2730}, 0, \frac{7}{6}, 0, -\frac{3617}{510}, 0, \ldots\right)$$
$$\vec{B}(2) = \left(1, \frac{3}{2}, \frac{13}{6}, 3, \frac{119}{30}, 5, \frac{253}{42}, 7, \frac{239}{30}, 9, \frac{665}{66}, 11, \frac{32069}{2730}, 13, \frac{91}{6}, 15, \ldots\right)$$
$$\vec{B}(3) = \left(1, \frac{5}{2}, \frac{37}{6}, 15, \frac{1079}{30}, 85, \frac{8317}{42}, 455, \frac{30959}{30}, 2313, \frac{338585}{66}, 11275, \ldots\right)$$

where argument 0 corresponds to the ordinary Bernoulli numbers and argument 1 corresponds to its variant with only difference in the sign of the second element ($B_1=+½$). The components of the vector **B**(n) The components of the vector **B** (n) are the values of the Bernoulli polynomials of gradually increasing degree calculated in n. (for demonstrations and further information, refer here)

## 2.4 From the $T^hA^{-1}$ matrix to the ordinary generalized Faulhaber formula.

The $T^hA^{-1}$ matrix can be ordered by expressing it equivalently in one of the following two ways (for proofs, I refer here):

$$\begin{pmatrix} 1\frac{1}{1}B_0(h) & 0 & 0 & 0 & 0 & 0 \\ 1\frac{1}{1}B_1(h) & 1\frac{1}{2}B_0(h) & 0 & 0 & 0 & 0 \\ 1\frac{1}{1}B_2(h) & 2\frac{1}{2}B_1(h) & 1\frac{1}{3}B_0(h) & 0 & 0 & 0 \\ 1\frac{1}{1}B_3(h) & 3\frac{1}{2}B_2(h) & 3\frac{1}{3}B_1(h) & 1\frac{1}{4}B_0(h) & 0 & 0 \\ 1\frac{1}{1}B_4(h) & 4\frac{1}{2}B_3(h) & 6\frac{1}{3}B_2(h) & 4\frac{1}{4}B_1(h) & 1\frac{1}{5}B_0(h) & 0 \\ 1\frac{1}{1}B_5(h) & 5\frac{1}{2}B_4(h) & 10\frac{1}{3}B_3(h) & 10\frac{1}{4}B_2(h) & 5\frac{1}{5}B_1(h) & 1\frac{1}{6}B_0(h) \end{pmatrix}$$

$$\begin{pmatrix} 1\frac{1}{1}B_0(h) & 0 & 0 & 0 & 0 & 0 \\ 2\frac{1}{2}B_1(h) & 1\frac{1}{2}B_0(h) & 0 & 0 & 0 & 0 \\ 3\frac{1}{3}B_2(h) & 3\frac{1}{3}B_1(h) & 1\frac{1}{3}B_0(h) & 0 & 0 & 0 \\ 4\frac{1}{4}B_3(h) & 6\frac{1}{4}B_2(h) & 4\frac{1}{4}B_1(h) & 1\frac{1}{4}B_0(h) & 0 & 0 \\ 5\frac{1}{5}B_4(h) & 10\frac{1}{5}B_3(h) & 10\frac{1}{5}B_2(h) & 5\frac{1}{5}B_1(h) & 1\frac{1}{5}B_0(h) & 0 \\ 6\frac{1}{6}B_5(h) & 15\frac{1}{6}B_4(h) & 20\frac{1}{6}B_3(h) & 15\frac{1}{6}B_2(h) & 6\frac{1}{6}B_1(h) & 1\frac{1}{6}B_0(h) \end{pmatrix}$$

from the first, generalizing, we obtain:

$$\sum_{j=0}^{n-1}(j+h)^m = \sum_{k=1}^{m}\frac{1}{k}\binom{m-1}{k-1}B_{m-k}(h)n^k$$

from the second:

$$\sum_{j=0}^{n-1}(j+h)^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(h)n^{m+1-k}$$

where $B_k(h)$ are the Bernoulli polynomials as a function of h corresponding to the first column of the matrix.

$B_k(0)=B_k$ where $B_k(h)$ are the Bernoulli polynomials as a function of h corresponding to the first column of the matrix.

$B_k(0) = B_k$ are the ordinary Bernoulli numbers, $B_k(1) = B_k^+$ are the Bernoulli numbers in the variant with B=+½ Particularized for j =1 the second of these equations is the formula called Faulhaber. For this formula often instead of using the sequence with $B_1 = +½$ on it uses the ordinary one and for this a factor $(-1)^k$ is added with the aim of changing sign to Bernoulli numbers with odd index of which the only non-zero it is $B_1 = +½$ which then becomes $B_1 = -½$

**2.5 Further generalization extended to any arithmetic progression.**

More generally we have:

$$\begin{pmatrix} 1 \\ h+ri \\ h^2+2hri+r^2i^2 \\ h^3+3h^2ri+3hr^2i^2+r^3i^3 \\ h^4+4h^3ri+6h^2r^2i^2+4hr^3i^3+r^4i^4 \\ h^5+5h^4ri+10h^3r^2i^2+10h^2r^3i^3+5hr^4i^4+r^5i^5 \end{pmatrix} = \begin{pmatrix} (h+ri)^0 \\ (h+ri)^1 \\ (h+ri)^2 \\ (h+ri)^3 \\ (h+ri)^4 \\ (h+ri)^5 \end{pmatrix}$$

equivalent to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h & r & 0 & 0 & 0 & 0 \\ h^2 & 2hr & r^2 & 0 & 0 & 0 \\ h^3 & 3h^2r & 3hr^2 & r^3 & 0 & 0 \\ h^4 & 4h^3r & 6h^2r^2 & 4hr^3 & r^4 & 0 \\ h^5 & 5h^4r & 10h^3r^2 & 10h^2r^3 & 5hr^4 & r^5 \end{pmatrix} \begin{pmatrix} 1 \\ i \\ i^2 \\ i^3 \\ i^4 \\ i^5 \end{pmatrix} = \begin{pmatrix} (h+ri)^0 \\ (h+ri)^1 \\ (h+ri)^2 \\ (h+ri)^3 \\ (h+ri)^4 \\ (h+ri)^5 \end{pmatrix}$$

expressible as:

$$T_{h,r}\vec{V}(i) = \vec{V}(h+ri)$$

as we see the multiplication by the $T_{h,r}$ matrix transforms the base of the power vector linearly. So the set of these matrices forms a non-commutative group isomorphic to that of the composition of linear functions with a variable.

We therefore have:

$$T_{a,b}T_{c,d} = T_{a+bc,bd}$$

$$T_{a,b}T_{a,b}^{-1} = T_{a,b}T_{-a/b,1/b} = T_{0,1} = I$$

$$T_{h,1} = T^h \ , \qquad T_{1,1} = T \ , \qquad T_{0,1} = T^0 = I$$

case r = 1 identifies the commutative subgroup of the powers of T isomorphic to the ordinary sum already examined.

## 2.6) Faulhaber's formula extended to any arithmetic progression

As seen on the matrix $T_{h,r}$ and for the rules of matrix algebra we have:

$$\vec{S}_{h,r}(n) = \begin{pmatrix} \sum_{i=0}^{n-1}(h+ri)^0 \\ \sum_{i=0}^{n-1}(h+ri)^1 \\ \sum_{i=0}^{n-1}(h+ri)^2 \\ \sum_{i=0}^{n-1}(h+ri)^3 \\ \sum_{i=0}^{n-1}(h+ri)^4 \\ \sum_{i=0}^{n-1}(h+ri)^5 \end{pmatrix} = \sum_{i=0}^{n-1}\vec{V}(h+ri) = \sum_{i=0}^{n-1}T_{h.r}\vec{V}(i) = T_{h.r}\sum_{i=0}^{n-1}\vec{V}(i)$$

and therefore by theorem 1B:

$$\vec{S}_{h,r}(n) = T_{h,r}A^{-1}n\vec{V}(n) \qquad (2)$$

The same formula without matrices is equivalent to:

$$\sum_{k=0}^{n-1}(h+rk)^m = \sum_{j=0}^{m} n^{j+1} \sum_{k=j}^{m} \binom{m}{k}\binom{k}{j} \frac{h^{m-k} r^k}{1+j} B_{k-j}$$

Example in the case m = 6
the matrices to be multiplied to obtain the polynomial coefficients are:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ h & r & 0 & 0 & 0 & 0 \\ h^2 & 2hr & r^2 & 0 & 0 & 0 \\ h^3 & 3h^2r & 3hr^2 & r^3 & 0 & 0 \\ h^4 & 4h^3r & 6h^2r^2 & 4hr^3 & r^4 & 0 \\ h^5 & 5h^4r & 10h^3r^2 & 10h^2r^3 & 5hr^4 & r^5 \end{pmatrix}$$

$$\begin{pmatrix} 1\frac{1}{1}B_0 & 0 & 0 & 0 & 0 & 0 \\ 1\frac{1}{1}B_1 & 1\frac{1}{2}B_0 & 0 & 0 & 0 & 0 \\ 1\frac{1}{1}B_2 & 2\frac{1}{2}B_1 & 1\frac{1}{3}B_0 & 0 & 0 & 0 \\ 1\frac{1}{1}B_3 & 3\frac{1}{2}B_2 & 3\frac{1}{3}B_1 & 1\frac{1}{4}B_0 & 0 & 0 \\ 1\frac{1}{1}B_4 & 4\frac{1}{2}B_3 & 6\frac{1}{3}B_2 & 4\frac{1}{4}B_1 & 1\frac{1}{5}B_0 & 0 \\ 1\frac{1}{1}B_5 & 5\frac{1}{2}B_4 & 10\frac{1}{3}B_3 & 10\frac{1}{4}B_2 & 5\frac{1}{5}B_1 & 1\frac{1}{6}B_0 \end{pmatrix}$$

## 2.7) Retouching the Faulhaber formula to extend it to any arithmetic progression

Since $T_{h,r}=T_{0,r}T_{h/r,1}$ we can write the (1) as:

$$\vec{S}_{h,r}(n) = T_{0,r}T_{\frac{h}{r},1}A^{-1}n\vec{V}(n)$$

from witch:

$$T_{0,r}\begin{pmatrix} 1\frac{1}{1}B_0(\frac{h}{r}) & 0 & 0 & 0 & 0 & 0 \\ 2\frac{1}{2}B_1(\frac{h}{r}) & 1\frac{1}{2}B_0(\frac{h}{r}) & 0 & 0 & 0 & 0 \\ 3\frac{1}{3}B_2(\frac{h}{r}) & 3\frac{1}{3}B_1(\frac{h}{r}) & 1\frac{1}{3}B_0(\frac{h}{r}) & 0 & 0 & 0 \\ 4\frac{1}{4}B_3(\frac{h}{r}) & 6\frac{1}{4}B_2(\frac{h}{r}) & 4\frac{1}{4}B_1(\frac{h}{r}) & 1\frac{1}{4}B_0(\frac{h}{r}) & 0 & 0 \\ 5\frac{1}{5}B_4(\frac{h}{r}) & 10\frac{1}{5}B_3(\frac{h}{r}) & 10\frac{1}{5}B_2(\frac{h}{r}) & 5\frac{1}{5}B_1(\frac{h}{r}) & 1\frac{1}{5}B_0(\frac{h}{r}) & 0 \\ 6\frac{1}{6}B_5(\frac{h}{r}) & 15\frac{1}{6}B_4(\frac{h}{r}) & 20\frac{1}{6}B_3(\frac{h}{r}) & 15\frac{1}{6}B_2(\frac{h}{r}) & 6\frac{1}{6}B_1(\frac{h}{r}) & 1\frac{1}{6}B_0(\frac{h}{r}) \end{pmatrix} =$$

$$= \begin{pmatrix} 1\frac{1}{1}B_0(\frac{h}{r}) & 0 & 0 & 0 & 0 & 0 \\ 2\frac{r}{2}B_1(\frac{h}{r}) & 1\frac{r}{2}B_0(\frac{h}{r}) & 0 & 0 & 0 & 0 \\ 3\frac{r^2}{3}B_2(\frac{h}{r}) & 3\frac{r^2}{3}B_1(\frac{h}{r}) & 1\frac{r^2}{3}B_0(\frac{h}{r}) & 0 & 0 & 0 \\ 4\frac{r^3}{4}B_3(\frac{h}{r}) & 6\frac{r^3}{4}B_2(\frac{h}{r}) & 4\frac{r^3}{4}B_1(\frac{h}{r}) & 1\frac{r^3}{4}B_0(\frac{h}{r}) & 0 & 0 \\ 5\frac{r^4}{5}B_4(\frac{h}{r}) & 10\frac{r^4}{5}B_3(\frac{h}{r}) & 10\frac{r^4}{5}B_2(\frac{h}{r}) & 5\frac{r^4}{5}B_1(\frac{h}{r}) & 1\frac{r^4}{5}B_0(\frac{h}{r}) & 0 \\ 6\frac{r^5}{6}B_5(\frac{h}{r}) & 15\frac{r^5}{6}B_4(\frac{h}{r}) & 20\frac{r^5}{6}B_3(\frac{h}{r}) & 15\frac{r^5}{6}B_2(\frac{h}{r}) & 6\frac{r^5}{6}B_1(\frac{h}{r}) & 1\frac{r^5}{6}B_0(\frac{h}{r}) \end{pmatrix}$$

this result, without matrices, can be expressed as:

$$\sum_{k=0}^{n-1}(h+rk)^m = \frac{r^m}{m+1}\sum_{k=0}^{m}\binom{m+1}{k+1}B_{m-k}(\frac{h}{r})n^{k+1} \qquad (3)$$

formula that extends Faulhaber's formula to any arithmetic progression which, in the particular case **h=0 e r=1,** is:

$$\sum_{k=0}^{n-1} k^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k+1}B_{m-k}n^{k+1}$$

the most common is the particular case h=1 r=1:

$$\sum_{k=0}^{n-1}(1+k)^m = \sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}B_{m-k}(1)n^{k+1}$$

as it is known the Bernoulli polynomials calculated in 1 almost identical to the same calculated in 0, which are the Bernoulli numbers, often the previous one is written:

$$\sum_{k=1}^{n} k^m = \sum_{k=0}^{m}\frac{1}{k+1}\binom{m}{k}B^{+}_{m-k}n^{k+1}$$

where it is:

$$B^{+}_{j} = B_j(1)$$

variant of Bernoulli numbers with the only difference:

$$B_1^+ = -B_1(0) = -B_1 = \frac{1}{2}$$

**2.8 Umbral calculus and translational properties**

With the notation of the umbral calculation, assimilating the index of the polynomial to an exponent, the previous one (3) can be written:

$$S_{h,r}^m(n) = \frac{r^m}{m+1}((B(\frac{h}{r}) + n)^{m+1} - B_{m+1}(\frac{h}{r}))$$

for the translational property of the polynomials of Bernoulli (and more generally of those of Appell) it results:

$$(B(\frac{h}{r}) + n)^{m+1} = B_{m+1}(\frac{h}{r} + n)$$

so:

$$S_{h,r}^m(n) = \sum_{k=0}^{n-1}(h + kr)^m = \frac{r^m}{m+1}(B_{m+1}(\frac{h}{r} + n) - B_{m+1}(\frac{h}{r}))$$

*Note*

After my discovery, looking on the net, I found that, a few years earlier, Bazso and Mezo had reached my same result:

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# On the coefficients of power sums of arithmetic progressions

András Bazsó [a] and István Mező [b]

[a]Institute of Mathematics, MTA-DE Research Group "Equations, Functions and Curves", Hungarian Academy of Sciences and University of Debrecen, P.O. Box 12, H-4010 Debrecen, Hungary

[b]Department of Mathematics, Nanjing University of Information Science and Technology, Nanjing, 210044 China

Recently, Bazsó et al. [1] considered the more general power sum

$$S_{m,r}^{n}(\ell) = r^n + (m+r)^n + (2m+r)^n + \cdots + ((\ell-1)m+r)^n,$$

where $m \neq 0, r$ are coprime integers. Obviously, $S_{1,0}^{n}(\ell) = S_n(\ell)$. They, among other things, proved that $S_{m,r}^{n}(\ell)$ is a polynomial of $\ell$ with the explicit expression

$$S_{m,r}^{n}(\ell) = \frac{m^n}{n+1}\left(B_{n+1}\left(\ell + \frac{r}{m}\right) - B_{n+1}\left(\frac{r}{m}\right)\right). \tag{2}$$

In [12], using a different approach, Howard also obtained the above relation via generating functions. Hirschhorn [11] and Chapman [8] deduced a longer expression which contains already just binomial coefficients and Bernoulli numbers.